LECTURE 22: The Poisson process

- Definition of the Poisson process
  - applications
- Distribution of number of arrivals
- The time of the kth arrival
- Memorylessness
- Distribution of interarrival times

Definition of the Poisson process

Poisson

$$\sum_{k=0}^{\infty} P(k,\tau) = 1$$

- Numbers of arrivals in disjoint time intervals are **independent**

$P(k,\tau)$ = Prob. of k arrivals in interval of duration τ

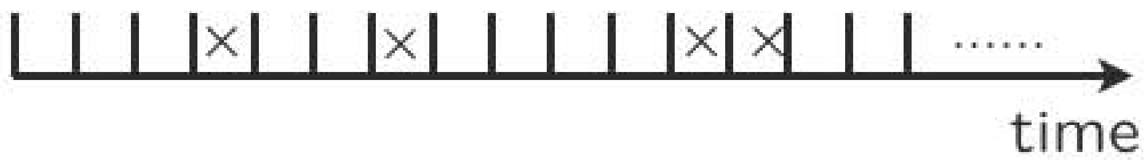
- **Small interval probabilities:**
  For VERY small δ:

$$P(k,\delta) \approx \begin{cases} 1-\lambda\delta & \text{if } k=0 \\ \lambda\delta & \text{if } k=1 \\ 0 & \text{if } k>1 \end{cases} \qquad P(k,\delta) = \begin{cases} 1-\lambda\delta + O(\delta^2) & \text{if } k=0 \\ \lambda\delta + O(\delta^2) & \text{if } k=1 \\ 0 + O(\delta^2) & \text{if } k>1 \end{cases}$$

$$\frac{O(\delta^2)}{\delta} \xrightarrow[\delta\to 0]{} 0$$

λ: "arrival rate"

Bernoulli



- Independence
- **Time homogeneity:**
  Constant p at each slot

Applications of the Poisson process

- Deaths from horse kicks in the Prussian army (1898)
- Particle emissions and radioactive decay
- Photon arrivals from a weak source
- Financial market shocks
- Placement of phone calls, service requests, etc.

Siméon Denis Poisson
(1781-1840)

The Poisson PMF for the number of arrivals

$\mathbf{P}(k, \delta)$

- N_τ: arrivals in $[0, \tau]$ $\quad P(k,\tau) = \mathbf{P}(N_\tau = k)$

$n = \tau/\delta$ intervals/slots of length δ ← small

$\mathbf{P}$(some slot contains two or more arrivals)

$\leq \sum_i \mathbf{P}$(slot i has ≥ 2 arrivals)

$= \frac{\tau}{\delta} O(\delta^2) \xrightarrow[\delta \to 0]{} 0$

$\mathbf{P}$(k arrivals in Poisson) $\approx \mathbf{P}$(k slots have arrival)

$N_\tau \approx$ binomial $\qquad p = \lambda\delta + O(\delta^2)$

$np = \lambda\tau + O(\delta) \approx \lambda\tau$

Bernoulli

$$p_S(k) = \frac{n!}{(n-k)!\,k!} \cdot p^k (1-p)^{n-k},$$

$$k = 0, \ldots, n$$

$\lambda = np \qquad n \to \infty \qquad p \to 0$

For fixed $k = 0, 1, \ldots,$

$$p_S(k) \to \frac{\lambda^k}{k!} e^{-\lambda},$$

$$P(k,\tau) = \frac{(\lambda\tau)^k e^{-\lambda\tau}}{k!}, \qquad k = 0, 1, \ldots$$

Mean and variance of the number of arrivals

$$P(k,\tau) = \mathbf{P}(N_\tau = k) = \frac{(\lambda\tau)^k e^{-\lambda\tau}}{k!}, \qquad k = 0, 1, \ldots$$

$$\boxed{\begin{aligned} \mathbf{E}[N_\tau] &= \lambda\tau \\ \mathrm{var}(N_\tau) &= \lambda\tau \end{aligned}}$$

$$\mathbf{E}[N_\tau] = \sum_{k=0}^{\infty} k \frac{(\lambda\tau)^k e^{-\lambda\tau}}{k!} = \cdots = \lambda\tau$$

$$\lambda = \frac{E[N_\tau]}{\tau}$$

$N_\tau \approx \text{Binomial}(n,p)$

$n = \tau/\delta, \quad p = \lambda\delta + O(\delta^2)$

$$E[N_\tau] \approx np \approx \lambda\tau$$

$$\mathrm{var}(N_\tau) \approx np(1-p) \approx \lambda\tau$$

Example

- You get email according to a Poisson process, at a rate of $\lambda = 5$ messages per hour.

$$\mathbf{E}[N_\tau] = \lambda\tau$$

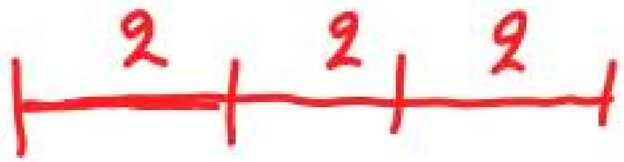
$$\mathrm{var}(N_\tau) = \lambda\tau$$

- Mean and variance of mails received during a day = $5 \cdot 24$

- $\mathbf{P}$(one new message in the next hour) = $P(1,1) = 5e^{-5}$

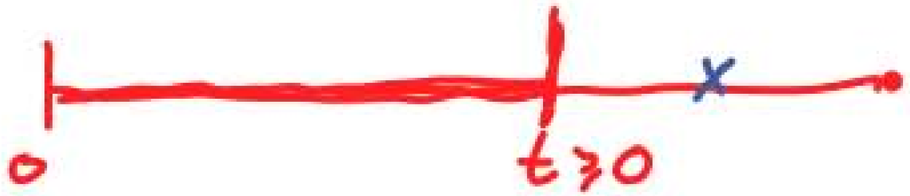
$$P(k,\tau) = \frac{(\lambda\tau)^k e^{-\lambda\tau}}{k!}, \qquad k = 0, 1, \ldots$$

- $\mathbf{P}$(exactly two messages during each of the next three hours) =



$$\left(P(2,1)\right)^3 = \left(\frac{5^2 e^{-5}}{2}\right)^3.$$

The time T_1 until the first arrival



$$P(k,\tau) = \frac{(\lambda\tau)^k e^{-\lambda\tau}}{k!}, \qquad k = 0, 1, \ldots$$

- Find the CDF: $\mathbf{P}(T_1 \le t) =$

$$= 1 - \mathbf{P}(T_1 > t) = 1 - P(0,t) = 1 - e^{-\lambda t}$$

$f_{T_1}(t) = \lambda e^{-\lambda t}, \quad$ for $t \ge 0$

Exponential(λ)

Memorylessness: conditioned on $T_1 > t$,
the PDF of $T_1 - t$ is again exponential

The time Y_k of the kth arrival

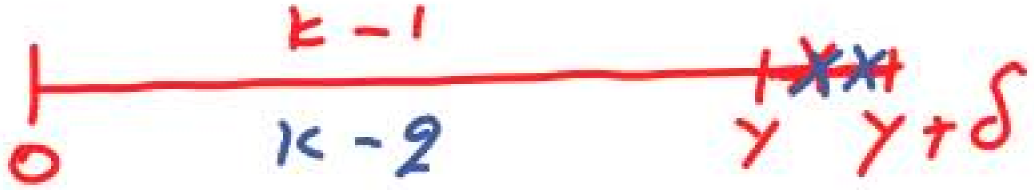
$$P(k,\tau) = \frac{(\lambda\tau)^k e^{-\lambda\tau}}{k!}, \qquad k = 0, 1, \ldots$$

- Can derive its PDF by first finding the CDF $P(Y_k \le y) = \sum_{n=k}^{\infty} P(n, y)$

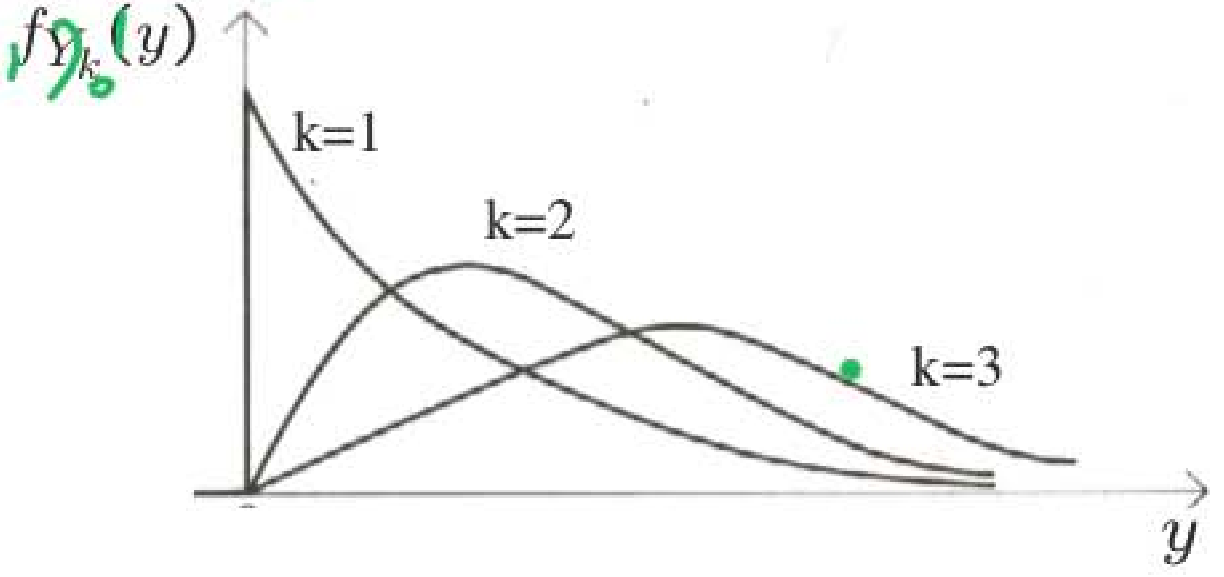
- More intuitive argument:

$$\boxed{f_{Y_k}(y)\cancel{\delta}} \approx \mathbf{P}(y \le Y_k \le y+\delta) =$$

$$\approx \boxed{P(k-1, y)\,\lambda\cancel{\delta}}$$
$$+ P(k-2, y)\, \cancel{O(\delta^2)}$$
$$+ \cancel{P(k-3, y)}\, O(\delta^2)$$



$$\frac{(\lambda y)^{k-1} e^{-\lambda y}}{(k-1)!}\lambda$$



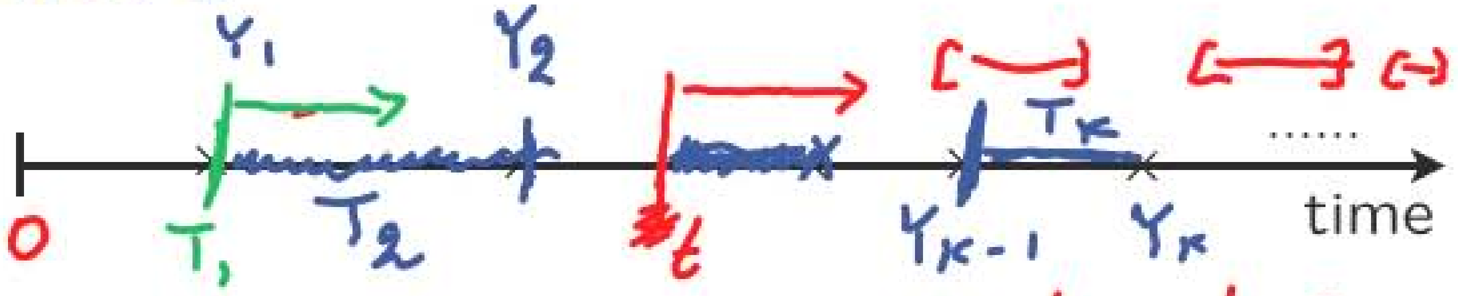
Erlang distribution: $f_{Y_k}(y) = \dfrac{\lambda^k y^{k-1} e^{-\lambda y}}{(k-1)!}, \quad y \ge 0$

order k

Memorylessness and the fresh-start property

- Analogous to the properties for the Bernoulli process
  - plausible, given the relation between the two processes
  - use intuitive reasoning
  - can be proved rigorously

Memorylessness and the fresh-start property



- If we start watching at time t,

  we see Poisson process, independent of the history until time t start fresh

  – time until next arrival: Exp(λ), independent of past

- If we start watching at time T_1, $T_1 = 3$

  we see Poisson process, independent of the history until time T_1

  – hence: time between first and second arrival, $T_2 = Y_2 - Y_1$ is: Exp(λ) ind. of T_1

  – similarly for all $T_k = Y_k - Y_{k-1}$, $k \geq 2$

$Y_k = T_1 + \cdots + T_k$ is sum of i.i.d. exponentials

$\mathbf{E}[Y_k] = k/\lambda \qquad \text{var}(Y_k) = k/\lambda^2$

- An equivalent definition
- A simulation method

Bernoulli/Poisson relation

$$n = \tau/\delta,$$

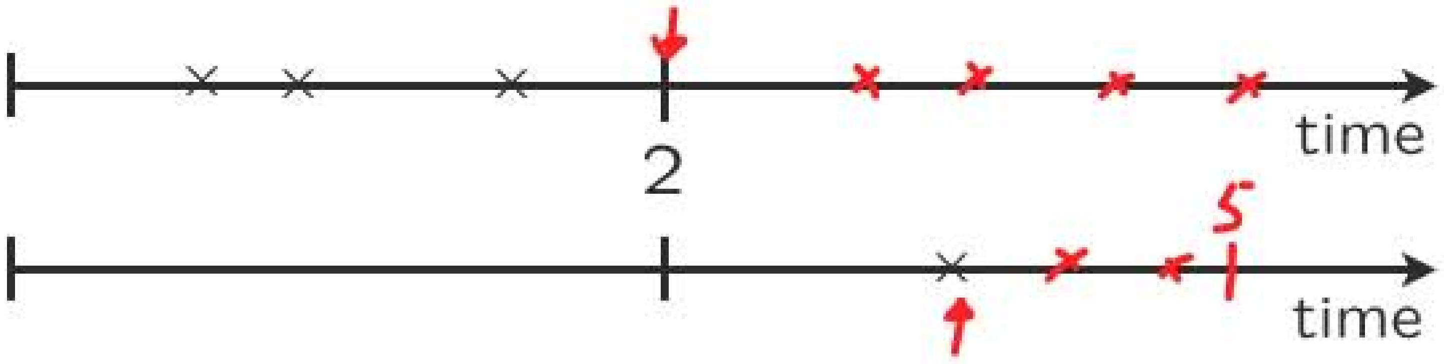
$$p = \lambda\delta$$

$$np = \lambda\tau$$

	POISSON	BERNOULLI
Times of Arrival	Continuous	Discrete
Arrival Rate	λ/unit time	p/per trial
PMF of # of Arrivals	Poisson	Binomial
Interarrival Time Distr.	Exponential	Geometric
Time to k-th arrival	Erlang	Pascal

Example: Poisson fishing

- Fish are caught as a Poisson process, $\lambda = 0.6$/hour
  - fish for two hours;
  - if you caught at least one fish, stop
  - else continue until first fish is caught



$\mathbf{P}$(fish for more than two hours)= $P(0,2)$

$$P(T_1 > 2) = \int_2^\infty f_{T_1}(t)\,dt$$

$\mathbf{P}$(fish for more than two and less than five hours)=

$$P(0,2)\,(1 - P(0,3))$$

$$P(2 < T_1 \le 5) = \int_2^5 f_{T_1}(t)\,dt$$

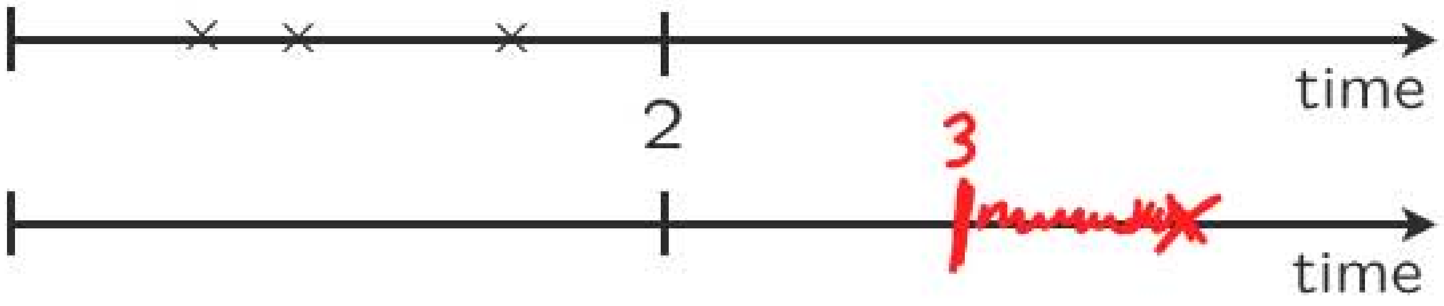
$$P(k,\tau) = \frac{(\lambda\tau)^k e^{-\lambda\tau}}{k!}$$

$$\mathbf{E}[N_\tau] = \lambda\tau$$

$$f_{Y_k}(y) = \frac{\lambda^k y^{k-1} e^{-\lambda y}}{(k-1)!}$$

Example: Poisson fishing

- Fish are caught as a Poisson process, $\lambda = 0.6/\text{hour}$
  - fish for two hours;
  - if you caught at least one fish, stop
  - else continue until first fish is caught



$\mathbf{P}$(catch at least two fish)=

$$\sum_{k=2}^{\infty} P(k,2) = 1 - P(0,2) - P(1,2)$$

$$P(Y_2 \le 2) = \int_0^2 f_{Y_2}(y)\,dy$$

$\mathbf{E}$[future fishing time | already fished for three hours]= $\frac{1}{\lambda}$

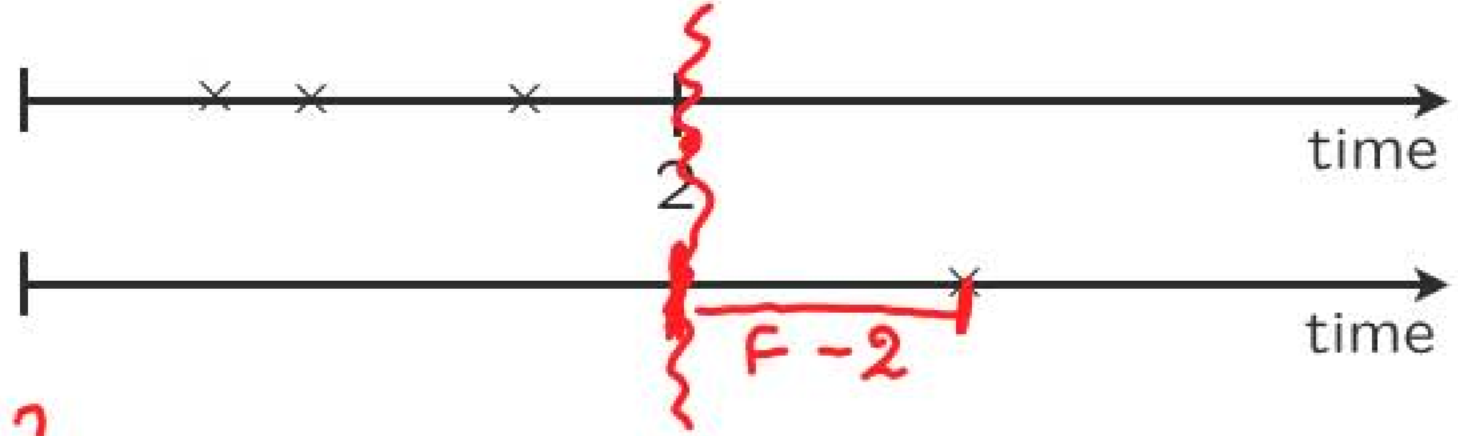
$$P(k,\tau) = \frac{(\lambda\tau)^k e^{-\lambda\tau}}{k!}$$

$$\mathbf{E}[N_\tau] = \lambda\tau$$

$$f_{Y_k}(y) = \frac{\lambda^k y^{k-1} e^{-\lambda y}}{(k-1)!}$$

Example: Poisson fishing

- Fish are caught as a Poisson process, $\lambda = 0.6$/hour
  - fish for two hours;
  - if you caught at least one fish, stop
  - else continue until first fish is caught



$\mathbf{E}$[total fishing time]= $E[F] = 2 + E[F-2]$

$= 2 + P(F=2)\cdot 0 + P(F>2)\, E[F-2 \mid F>2]$

$= 2 + P(0,2)\cdot 1/\lambda$

$\mathbf{E}$[number of fish]= $\underbrace{\lambda\tau}_{0.6 * 2} + P(0,2)\cdot 1$

$$P(k,\tau) = \frac{(\lambda\tau)^k e^{-\lambda\tau}}{k!}$$

$$\mathbf{E}[N_\tau] = \lambda\tau$$

$$f_{Y_k}(y) = \frac{\lambda^k y^{k-1} e^{-\lambda y}}{(k-1)!}$$