# LECTURE 7: Conditioning on a random variable; Independence of r.v.'s

- Conditional PMFs
  - Conditional expectations
  - Total expectation theorem
- Independence of r.v.'s
  - Expectation properties
  - Variance properties
- The variance of the binomial
- The hat problem: mean and variance

# Conditional PMFs

$A = \{Y = y\}$

$$p_{X|A}(x \mid A) = \mathbf{P}(X = x \mid A)$$

$$p_{X|Y}(x \mid y) = \mathbf{P}(X = x \mid Y = y) = \frac{\mathbf{P}(X = x, Y = y)}{\mathbf{P}(Y = y)}$$

$$\boxed{p_{X|Y}(x \mid y) = \frac{p_{X,Y}(x,y)}{p_Y(y)} \qquad \text{defined for } y \text{ such that } p_Y(y) > 0}$$

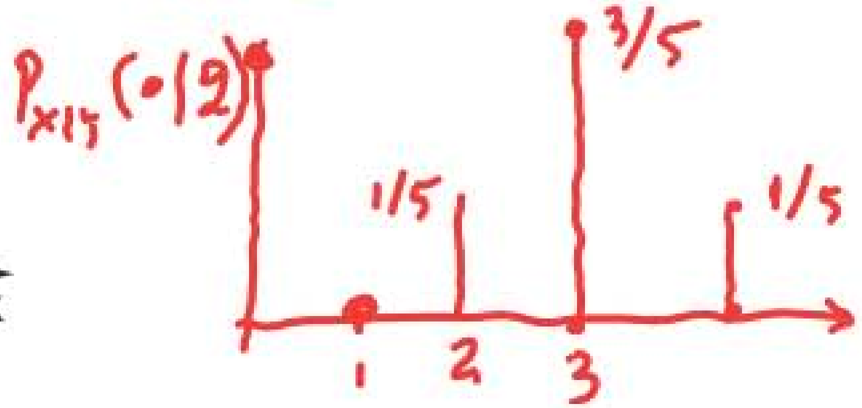
$$\sum_x p_{X|Y}(x \mid y) = 1$$

$Y = 2$ $\qquad p_Y(2) = 5/20$

$p_{X|Y}(1 \mid 2) = 0$

$p_{X|Y}(2 \mid 2) = 1/5$

| y \ x | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| 4 | 1/20 | 2/20 | 2/20 | |
| 3 | 2/20 | 4/20 | 1/20 | 2/20 |
| 2 | | 1/20 | 3/20 | 1/20 |
| 1 | | 1/20 | | |




$$p_{X,Y}(x,y) = p_Y(y)\, p_{X|Y}(x \mid y)$$

$$p_{X,Y}(x,y) = p_X(x)\, p_{Y|X}(y \mid x)$$

# Conditional PMFs involving more than two r.v.'s

- Self-explanatory notation

$$p_{X|Y,Z}(x \mid y, z) = P(X=x \mid Y=y, Z=z) = \frac{P(X=x, Y=y, Z=z)}{P(Y=y, Z=z)} = \frac{p_{X,Y,Z}(x,y,z)}{p_{Y,Z}(y,z)}$$

$$p_{X,Y|Z}(x, y \mid z) = P(X=x, Y=y \mid Z=z)$$

- Multiplication rule

$$\mathbf{P}(A \cap B \cap C) = \mathbf{P}(A)\,\mathbf{P}(B \mid A)\,\mathbf{P}(C \mid A \cap B)$$

$A = \{X=x\}$ $B = \{Y=y\}$ $C = \{Z=z\}$

$$p_{X,Y,Z}(x, y, z) = p_X(x)\, p_{Y|X}(y \mid x)\, p_{Z|X,Y}(z \mid x, y)$$

## Conditional expectation

$A = \{Y = y\}$

$$\mathbf{E}[X] = \sum_x x p_X(x) \qquad \mathbf{E}[X \mid A] = \sum_x x\, p_{X|A}(x) \qquad \mathbf{E}[X \mid Y = y] = \sum_x x p_{X|Y}(x \mid y)$$

- Expected value rule

$$\mathbf{E}[g(X)] = \sum_x g(x)\, p_X(x) \qquad \mathbf{E}[g(X) \mid A] = \sum_x g(x)\, p_{X|A}(x)$$

$$\mathbf{E}[g(X) \mid Y = y] = \sum_x g(x)\, p_{X|Y}(x \mid y)$$

# Total probability and expectation theorems

- $A_1, \ldots, A_n$: partition of $\Omega$ $\quad Y = \{y_1, \ldots, y_n\} \quad A_i = \{Y = y_i\}$

- $p_X(x) = \mathbf{P}(A_1)\, p_{X|A_1}(x) + \cdots + \mathbf{P}(A_n)\, p_{X|A_n}(x)$

$$p_X(x) = \sum_y p_Y(y)\, p_{X|Y}(x \mid y)$$

- $\mathbf{E}[X] = \mathbf{P}(A_1)\, \mathbf{E}[X \mid A_1] + \cdots + \mathbf{P}(A_n)\, \mathbf{E}[X \mid A_n]$

$$\boxed{\mathbf{E}[X] = \sum_y p_Y(y)\, \mathbf{E}[X \mid Y = y]}$$

- Fine print:
  Also valid when $Y$ is a discrete r.v. that ranges over an infinite set,
  as long as $\mathbf{E}[|X|] < \infty$

# Independence

- of two events: $\mathbf{P}(A \cap B) = \mathbf{P}(A) \cdot \mathbf{P}(B)$ $\qquad$ $\mathbf{P}(A \mid B) = \mathbf{P}(A)$

- of a r.v. and an event: $\mathbf{P}(\underline{X = x} \text{ and } \underline{A}) = \mathbf{P}(X = x) \cdot \mathbf{P}(A)$, for all $x$

$p_{X|A}(x) = p_X(x)$, for all $x$ $\qquad$ $\mathbf{P}(A \mid X = x) = \mathbf{P}(A)$, for all $x$

- of two r.v.'s: $\mathbf{P}(\underline{X = x} \text{ and } \underline{Y = y}) = \mathbf{P}(X = x) \cdot \mathbf{P}(Y = y)$, for all $x, y$

$p_{X,Y}(x, y) = p_X(x)\, p_Y(y)$, for all $x$, $y$

$p_{X|Y}(x|y) = p_X(x)$

$p_{Y|X}(y|x) = p_Y(y)$

> $X, Y, Z$ are **independent** if:
> $p_{X,Y,Z}(x, y, z) = p_X(x)\, p_Y(y)\, p_Z(z)$, for all $x, y, z$

## Example: independence and conditional independence

| y \ x | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| 4 | 1/20 | 2/20 | 2/20 | |
| 3 | 2/20 | 4/20 | 1/20 | 2/20 |
| 2 | | 1/20 | 3/20 | 1/20 |
| 1 | | 1/20 | | |

| | | |
|---|---|---|
| 1/3 | 1/9 | 2/9 |
| 2/3 | 2/9 | 4/9 |
| | 1/3 | 2/3 |

- Independent? No

$p_X(1) = 3/20$

$p_{X|Y}(1|1) = 0$

- What if we condition on $X \leq 2$ and $Y \geq 3$?

Yes

## Independence and expectations

- In general: $\mathbf{E}[g(X,Y)] \neq g\big(\mathbf{E}[X], \mathbf{E}[Y]\big)$

always true

- Exceptions: $\mathbf{E}[aX+b] = a\mathbf{E}[X]+b$ $\qquad \mathbf{E}[X+Y+Z] = \mathbf{E}[X]+\mathbf{E}[Y]+\mathbf{E}[Z]$

If $X$, $Y$ are **independent:** $\mathbf{E}[XY] = \mathbf{E}[X]\mathbf{E}[Y]$

$g(X)$ and $h(Y)$ are also independent: $\mathbf{E}[g(X)h(Y)] = \mathbf{E}[g(X)] \cdot \mathbf{E}[h(Y)]$

$E[g(X,Y)] \qquad g(x,y) = xy$

$= \sum_x \sum_y xy\, p_{X,Y}(x,y) = \sum_x \sum_y x y\, p_X(x) p_Y(y)$

$= \sum_x x\, p_X(x) \underbrace{\sum_y y\, p_Y(y)} = E[X]\, E[Y]$

## Independence and variances

- Always true: $\text{var}(aX) = a^2\text{var}(X)$  $\text{var}(X + a) = \text{var}(X)$

- In general: $\text{var}(X + Y) \neq \text{var}(X) + \text{var}(Y)$

If $X$, $Y$ are independent: $\text{var}(X + Y) = \text{var}(X) + \text{var}(Y)$

assume

$E[X] = E[Y] = 0$

$E[XY] = E[X]E[Y] = 0$

$\text{var}(X+Y) = E[(X+Y)^2] = E[X^2 + 2XY + Y^2]$

$= E[X^2] + 2E[XY] + E[Y^2] = \text{var}(X) + \text{var}(Y)$

- Examples:
  - If $X = Y$: $\text{var}(X + Y) =$ $\text{var}(2X) = 4\text{var}(X)$
  - If $X = -Y$: $\text{var}(X + Y) =$ $\text{var}(0) = 0$
  - If $X$, $Y$ independent: $\text{var}(X - 3Y) =$ $\text{var}(X) + \text{var}(-3Y) = \text{var}(X) + 9\text{var}(Y)$

# Variance of the binomial

- $X$: binomial with parameters $n$, $p$
  - number of successes in $n$ independent trials

$X_i = 1$ if $i$th trial is a success; (indicator variable) independent
$X_i = 0$ otherwise

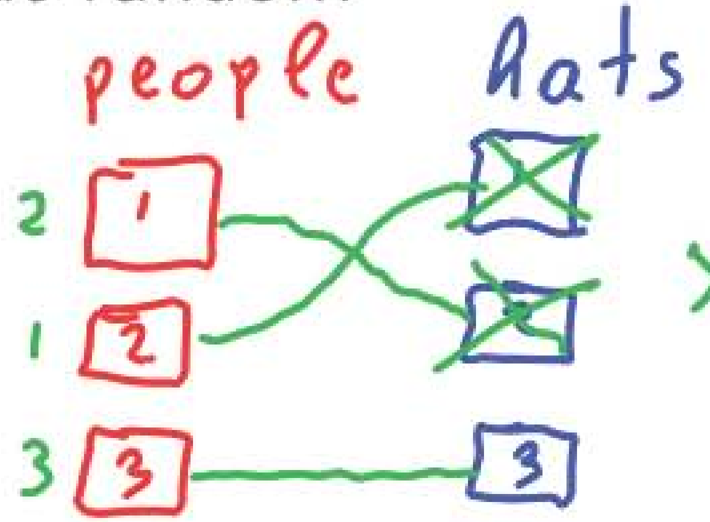
$X = X_1 + \cdots + X_n$

$\boxed{\mathrm{var}(X)} = \mathrm{var}(X_1) + \dots + \mathrm{var}(X_n)$

$= n \cdot \mathrm{var}(X_1) = \boxed{np(1-p)}$

## The hat problem

- $n$ people throw their hats in a box and then pick one at random
  - All permutations equally likely $\quad 1/n!$
  - Equivalent to picking one hat at a time




$X=1$

$\frac{1}{3}\cdot\frac{1}{2}\cdot 1 = \frac{1}{3!}$

$p_X(k)$: hard!

$\sum_k k\, p_X(k)$

- $X$: number of people who get their own hat
  - Find $\boxed{\mathbf{E}[X]} = E[X_1] + \cdots + E[X_n] = n\cdot\frac{1}{n} = \boxed{1}$

$$X_i = \begin{cases} 1, & \text{if } i \text{ selects own hat} \\ 0, & \text{otherwise.} \end{cases}$$

$$X = X_1 + X_2 + \cdots + X_n$$

- $\mathbf{E}[X_i] = E[X_1] = P(X_1 = 1) = \frac{1}{n}$

## The variance in the hat problem

- $X$: number of people who get their own hat
  - Find var$(X)$

$n = 2$

$X_1 = 1 \Rightarrow X_2 = 1$

$X_1 = 0 \Rightarrow X_2 = 0$

$$X_i = \begin{cases} 1, & \text{if } i \text{ selects own hat} \\ 0, & \text{otherwise.} \end{cases}$$

$$X = X_1 + X_2 + \cdots + X_n$$

$$X^2 = \underbrace{\sum_i X_i^2}_{n} + \underbrace{\sum_{i,j:i\neq j} X_i X_j}_{n^2 - n = n(n-1)}$$

- $\boxed{\text{var}(X)} = \mathbf{E}[X^2] - (\mathbf{E}[X])^2 = 2 - 1 = \boxed{1}$
- $\mathbf{E}[X_i^2] = E[X_1^2] = E[X_1] = 1/n$

$$E[X^2] = n \cdot \frac{1}{n} + n(n-1) \cdot \frac{1}{n} \cdot \frac{1}{n-1}$$

- For $i \neq j$: $\mathbf{E}[X_i X_j] = E[X_1 X_2] = P(X_1 X_2 = 1) = P(X_1 = 1, X_2 = 1)$

$$= P(X_1 = 1)\, P(X_2 = 1 \mid X_1 = 1) = \frac{1}{n} \cdot \frac{1}{n-1}$$