LECTURE 21: The Bernoulli process

- Definition of Bernoulli process
- Stochastic processes
- Basic properties (memorylessness)
- The time of the kth success/arrival
- Distribution of interarrival times
- Merging and splitting
- Poisson approximation

The Bernoulli process

- A sequence of independent Bernoulli trials, X_i
- At each trial, i:

$$\mathbf{P}(X_i = 1) = \mathbf{P}(\text{success at the } i\text{th trial}) = p$$
$$\mathbf{P}(X_i = 0) = \mathbf{P}(\text{failure at the } i\text{th trial}) = 1 - p$$

- Key assumptions:
  - Independence
  - Time-homogeneity
- Model of:
  - Sequence of lottery wins/losses
  - Arrivals (each second) to a bank
  - Arrivals (at each time slot) to server
  - ...

Jacob Bernoulli
(1655–1705)

Stochastic processes

- First view: sequence of random variables $X_1, X_2, \ldots$

  Interested in: $\mathbf{E}[X_i]$ var(X_i) $p_{X_i}(x)$

  $p_{X_1,\ldots,X_n}(x_1, \ldots, x_n)$

- Second view – sample space:

  $\Omega =$

- Example (for Bernoulli process):

  $\mathbf{P}(X_i = 1 \text{ for all } i) =$

Number of successes/arrivals S in n time slots

- $S =$

- $\mathbf{P}(S = k) =$

- $\mathbf{E}[S] =$

- $\mathrm{var}(S) =$

Time until the first success/arrival

- $T_1 =$

- $\mathbf{P}(T_1 = k) =$

- $\mathbf{E}[T_1] = \frac{1}{p}$

- $\text{var}(T_1) = \frac{1-p}{p^2}$

Independence, memorylessness, and fresh-start properties

- Fresh-start after time n

- Fresh-start after time T_1

Independence, memorylessness, and fresh-start properties

- Fresh-start after a random time N

  $N =$ time of 3rd success

  $N =$ first time that 3 successes in a row have been observed

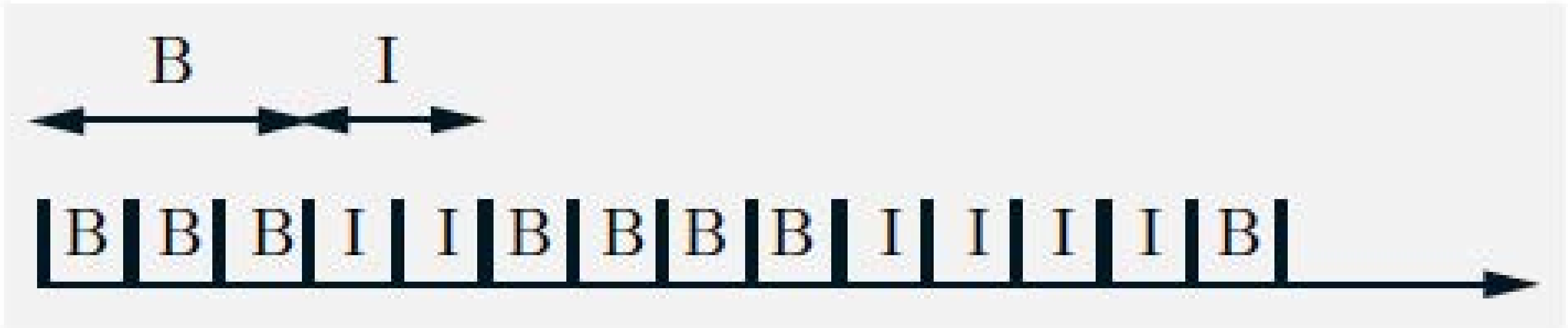
  $N =$ the time just before the first occurrence of 1,1,1

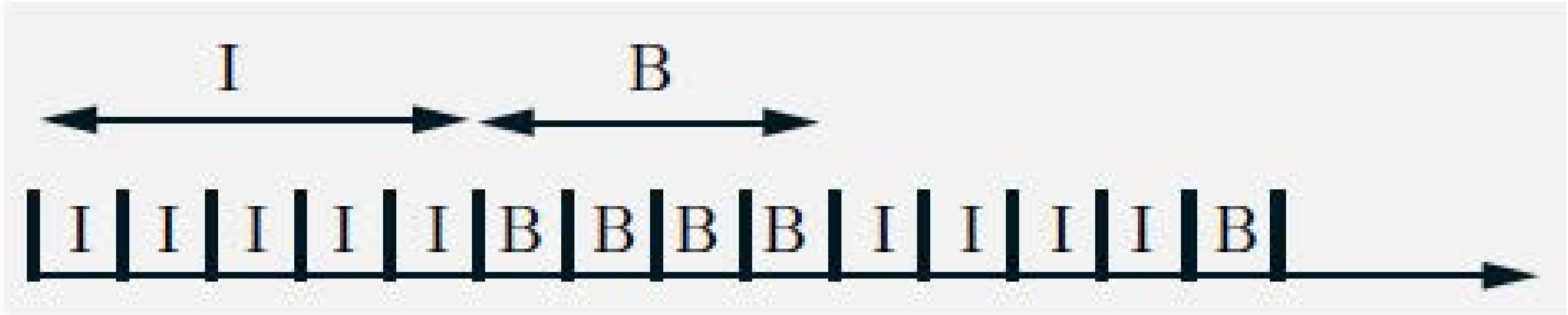
The process $X_{N+1}, X_{N+2}, \ldots$ is:
- a Bernoulli process
- independent of $N, X_1, \ldots, X_N$

(as long as N is determined "causally")

The distribution of busy periods

- At each slot, a server is busy or idle (Bernoulli process)
- First busy period:
  - starts with first busy slot
  - ends just before the first subsequent idle slot





Time of the kth success/arrival

- $Y_k =$ time of kth arrival

$Y_k = T_1 + \cdots + T_k$

- $T_k =$ kth inter-arrival time $= Y_k - Y_{k-1} \quad (k \geq 2)$

the T_i are i.i.d., Geometric(p)

- The process starts fresh after time T_1
- T_2 is independent of T_1; Geometric(p); etc.

Time of the kth success/arrival

$Y_k = T_1 + \cdots + T_k$

the T_i are i.i.d., Geometric(p)

$$\mathbf{E}[Y_k] = \frac{k}{p} \qquad \mathrm{var}(Y_k) = \frac{k(1-p)}{p^2}$$

$$p_{Y_k}(t) = \binom{t-1}{k-1} p^k (1-p)^{t-k},$$

$$t = k, k+1, \ldots$$

Merging of independent Bernoulli processes

Bernoulli(p)

merged process

Bernoulli($p + q - pq$)

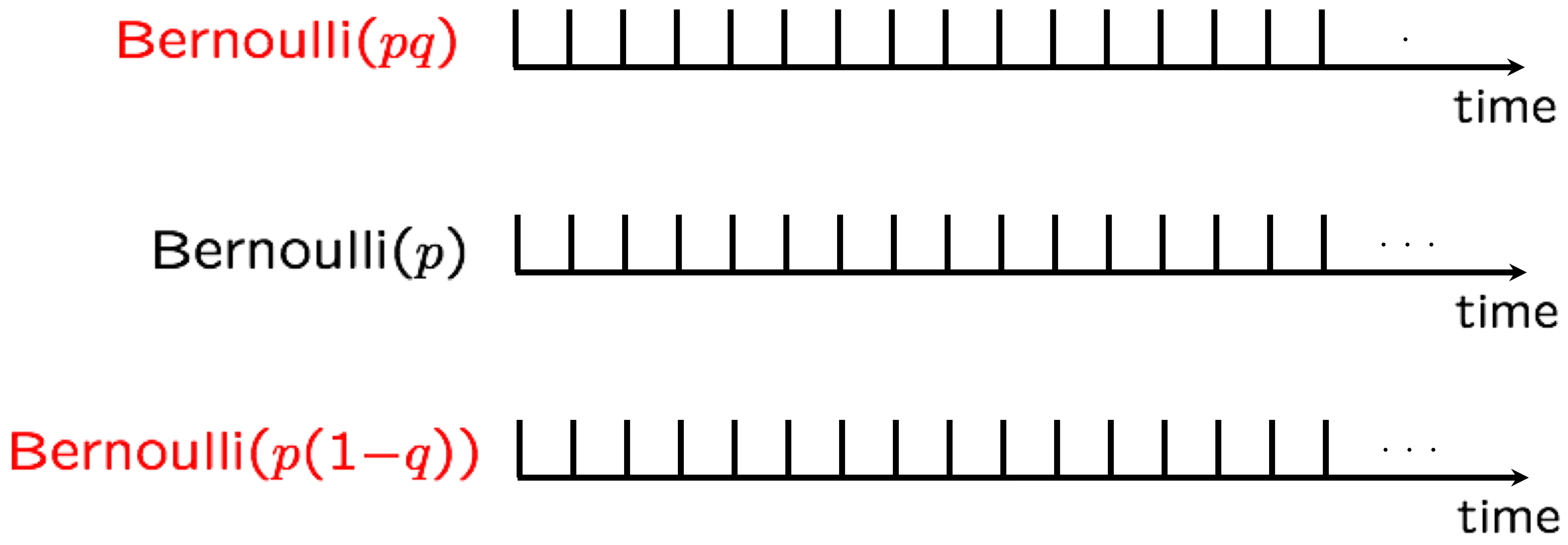
(collisions are counted as one arrival)

Bernoulli(q)

$\mathbf{P}$(arrival in first process | arrival) $=$

Splitting of a Bernoulli process

- Split successes into two streams, using independent flips of a coin with bias q
  - assume that coin flips are independent from the original Bernoulli process



- Are the two resulting streams independent?

Poisson approximation to binomial

- Interesting regime: large n , small p, moderate $\lambda = np$

- Number of arrivals S in n slots: $p_S(k) \;=\; \dfrac{n!}{(n-k)!\,k!} \cdot p^k(1-p)^{n-k}, \quad k = 0, \ldots, n$

For fixed $k = 0, 1, \ldots,$

$$p_S(k) \to \frac{\lambda^k}{k!} e^{-\lambda},$$

- Fact: for any fixed $k \geq 0$,
$\lim_{n\to\infty} (1 - \lambda/n)^{n-k} = e^{-\lambda}$