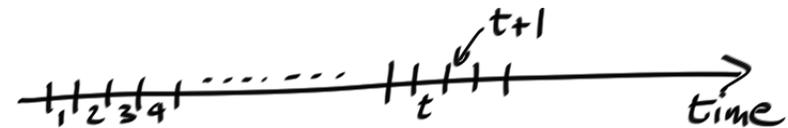


## Markov processes – I

- checkout counter example
- Markov process definition
- n-step transition probabilities
- classification of states

past future

STATE

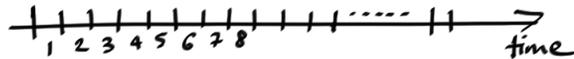


$$\text{state}(t+1) = f(\text{state}(t), \text{noise})$$

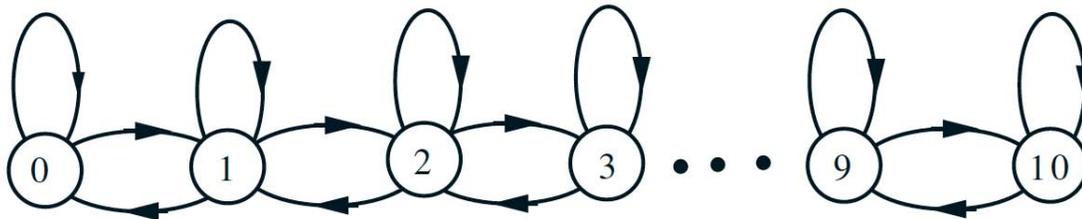
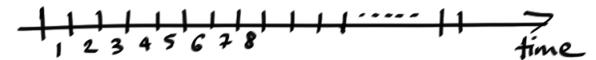
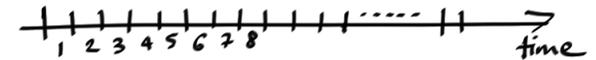
## checkout counter example



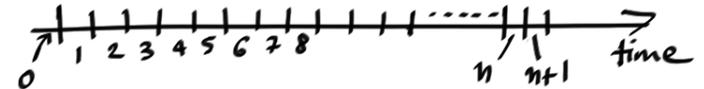
- discrete time  $n = 0, 1, \dots$



- customer arrivals: Bernoulli( $p$ )
- customer service times: geometric( $q$ )
- “state”  $X_n$ : number of customers at time  $n$



## discrete-time finite state Markov chains



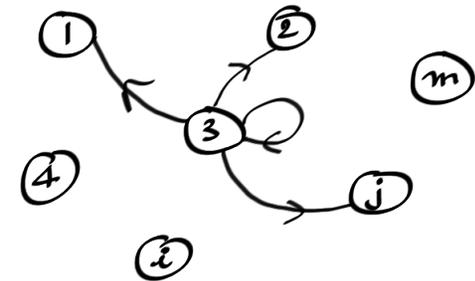
- $X_n$ : state after  $n$  transitions
  - belongs to a finite set
  - initial state  $X_0$  either given or random
  - transition probabilities:

$$\begin{aligned} p_{ij} &= \mathbf{P}(X_1 = j \mid X_0 = i) \\ &= \mathbf{P}(X_{n+1} = j \mid X_n = i) \end{aligned}$$

- Markov property/assumption:  
“given current state, the past doesn’t matter”

$$\begin{aligned} p_{ij} &= \mathbf{P}(X_{n+1} = j \mid X_n = i) \\ &= \mathbf{P}(X_{n+1} = j \mid X_n = i, X_{n-1}, \dots, X_0) \end{aligned}$$

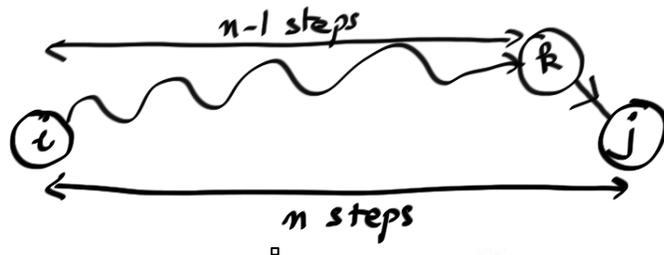
- model specification: identify states, transitions, and transition probabilities



## n-step transition probabilities

- state probabilities, given initial state  $i$ :

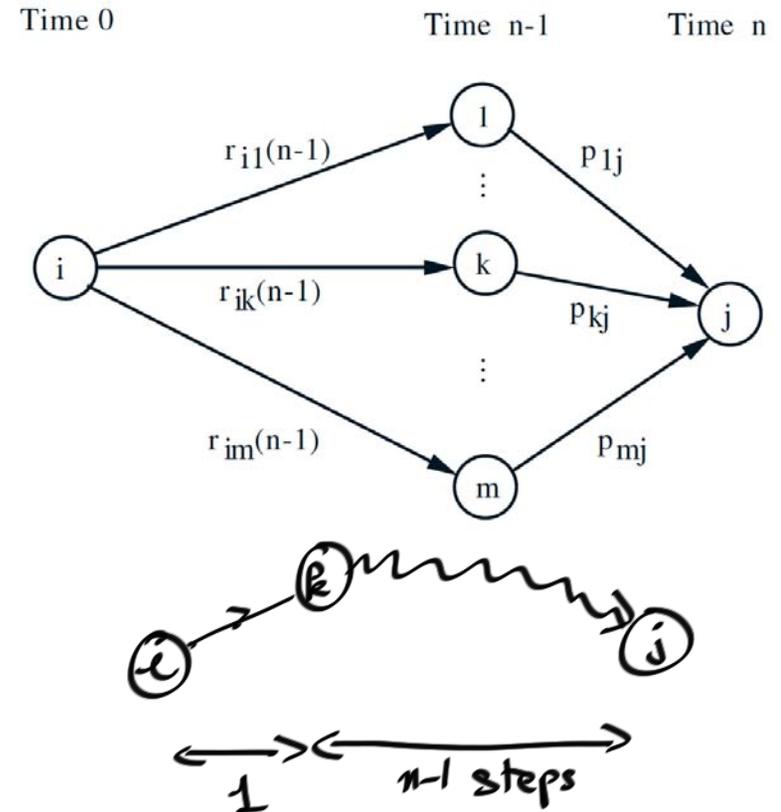
$$\begin{aligned} r_{ij}(n) &= \mathbf{P}(X_n = j \mid X_0 = i) \\ &= \mathbf{P}(X_{n+s} = j \mid X_s = i) \end{aligned}$$



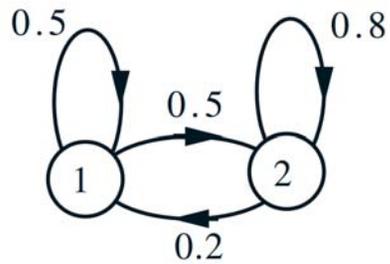
- key recursion: 
$$r_{ij}(n) = \sum_{k=1}^m r_{ik}(n-1)p_{kj}$$

- random initial state:

$$\mathbf{P}(X_n = j) = \sum_{i=1}^m \mathbf{P}(X_0 = i)r_{ij}(n)$$



## example

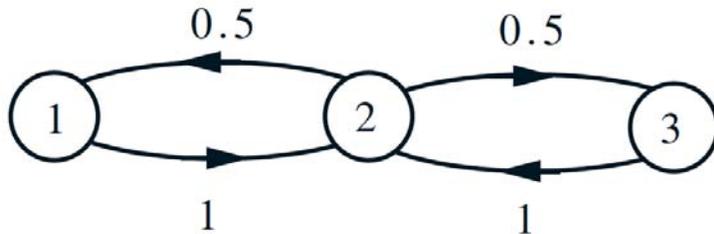


$$r_{ij}(n) = \mathbf{P}(X_n = j \mid X_0 = i)$$

	$n = 0$	$n = 1$	$n = 2$	$n = 100$	$n = 101$
$r_{11}(n)$					
$r_{12}(n)$					
$r_{21}(n)$					
$r_{22}(n)$					

## generic convergence questions

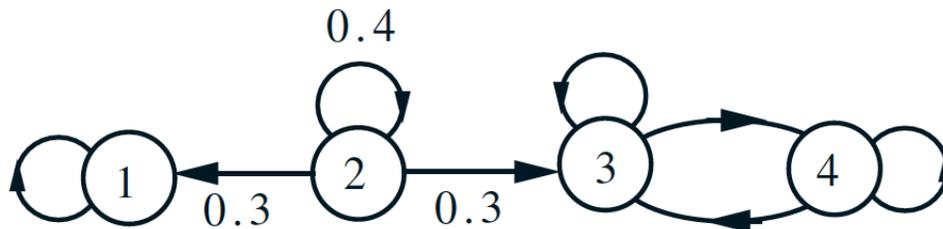
- does  $r_{ij}(n)$  converge to something?



$$n \text{ odd: } r_{22}(n) =$$

$$n \text{ even: } r_{22}(n) =$$

- does the limit depend on initial state?



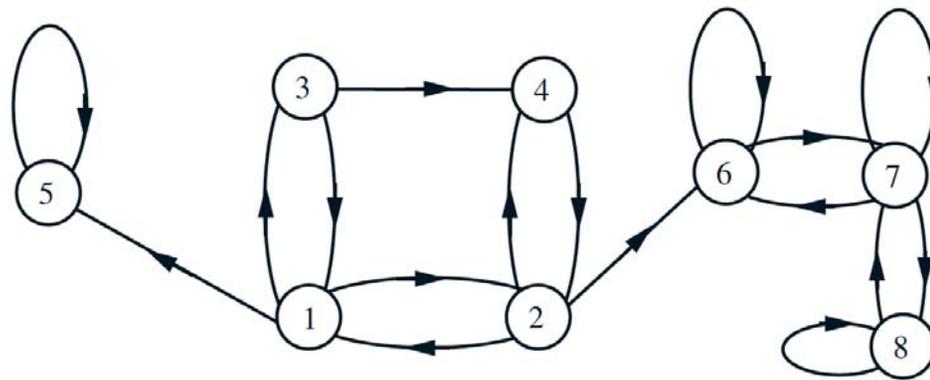
$$r_{11}(n) =$$

$$r_{31}(n) =$$

$$r_{21}(n) =$$

## recurrent and transient states

- state  $i$  is recurrent if “starting from  $i$ , and from wherever you can go, there is a way of returning to  $i$ ”
- if not recurrent, called transient



- recurrent class: a collection of recurrent states communicating only between each other

MIT OpenCourseWare  
<https://ocw.mit.edu>

**Resource: Introduction to Probability**  
John Tsitsiklis and Patrick Jaillet

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