LECTURE 10: Conditioning on a random variable; Independence; Bayes' rule

- Conditioning X on Y
  - Total probability theorem
  - Total expectation theorem

- Independence
  - independent normals

- A comprehensive example

- Four variants of the Bayes rule

Conditional PDFs, given another r.v.

$$p_{X|Y}(x \mid y) = \mathbf{P}(X = x \mid Y = y) = \frac{p_{X,Y}(x,y)}{p_Y(y)}, \quad \text{if } p_Y(y) > 0$$

$p_{X,Y}(x,y)$	$f_{X,Y}(x,y)$
$p_{X|A}(x)$	$f_{X|A}(x)$
$p_{X|Y}(x \mid y)$	$f_{X|Y}(x \mid y)$

Definition: $$f_{X|Y}(x \mid y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} \quad \text{if } f_Y(y) > 0$$

$$\mathbf{P}(x \le X \le x+\delta \mid \underbrace{A}_{Y=y \;\to\; Y\approx y}) \approx f_{X|A}(x) \cdot \delta, \qquad \text{where } \mathbf{P}(A) > 0$$

$$\mathbf{P}(x \le X \le x+\delta \mid y \le Y \le y+\epsilon) \approx \frac{f_{X,Y}(x,y)\,\delta\,\cancel{\epsilon}}{f_Y(y)\,\cancel{\epsilon}} = f_{X|Y}(x \mid y)\,\delta$$

Definition: $$\mathbf{P}(X \in A \mid Y = y) = \int_A f_{X\,|\,Y}(x \mid y)\,dx$$

Comments on conditional PDFs

$$f_{X|Y}(x \mid y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}$$

- $f_{X|Y}(x \mid y) \geq 0$

- Think of value of Y as fixed at some y

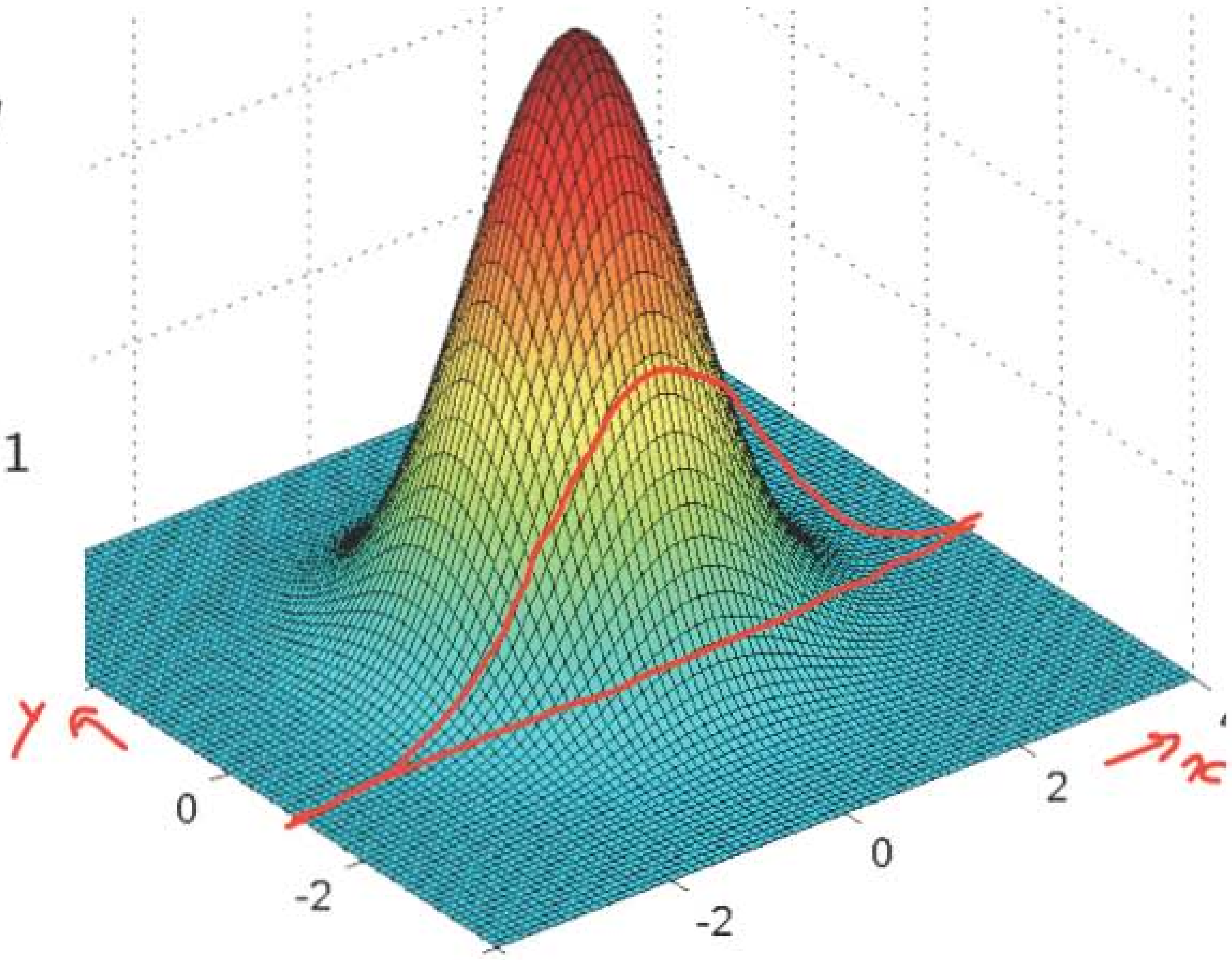
  shape of $f_{X|Y}(\cdot \mid y)$: slice of the joint

- $\displaystyle\int_{-\infty}^{\infty} f_{X|Y}(x \mid y)\,dx = \frac{\int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx}{f_Y(y)} = 1$

- Multiplication rule:

  $f_{X,Y}(x,y) = f_Y(y) \cdot f_{X|Y}(x \mid y)$

  $\qquad = f_X(x) \cdot f_{Y|X}(y \mid x)$



Total probability and expectation theorems

$$p_X(x) = \sum_y p_Y(y) p_{X|Y}(x \mid y)$$

$$f_X(x) = \int_{-\infty}^{\infty} \underbrace{f_Y(y) f_{X|Y}(x \mid y)}_{f_{X,Y}(x,y)} \, dy$$ TRm.

$$\mathbf{E}[X \mid Y = y] = \sum_x x p_{X|Y}(x \mid y)$$

$$\mathbf{E}[X \mid Y = y] = \int_{-\infty}^{\infty} x f_{X|Y}(x \mid y) \, dx$$ Def.

$$\mathbf{E}[X] = \sum_y p_Y(y) \mathbf{E}[X \mid Y = y]$$

$$\mathbf{E}[X] = \int_{-\infty}^{\infty} f_Y(y) \mathbf{E}[X \mid Y = y] \, dy$$

$$= \int_{-\infty}^{\infty} f_Y(y) \int_{-\infty}^{\infty} x f_{X|Y}(x|y) \, dx \, dy$$

$$= \int_{-\infty}^{\infty} x \underbrace{\int_{-\infty}^{\infty} f_Y(y) f_{X|Y}(x|y) \, dy}\, dx$$

$$= \int_{-\infty}^{\infty} x f_X(x) \, dx = E[X]$$

- Expected value rule...

$$E[g(X) \mid Y = y]$$

$$= \int_{-\infty}^{\infty} g(x) f_{X|Y}(x|y) \, dx$$

Independence

$$p_{X,Y}(x,y) = p_X(x)\, p_Y(y), \quad \text{for all } x,\ y$$

$$\boxed{f_{X,Y}(x,y) = \underline{f_X(x)}\, f_Y(y), \quad \text{for all } x \text{ and } y}$$

$f_{Y|X} = f_Y$

$$f_{X,Y}(x,y) = \underline{f_{X|Y}(x\,|\,y)}\, f_Y(y)$$

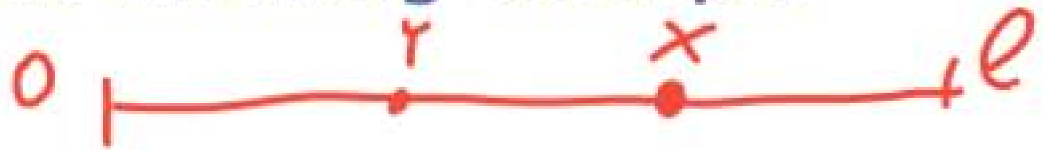
- equivalent to: $f_{X|Y}(x\,|\,y) = f_X(x)$, for all y with $f_Y(y) > 0$ and all x

If X, Y are **independent:** $\mathbf{E}[XY] = \mathbf{E}[X]\mathbf{E}[Y]$

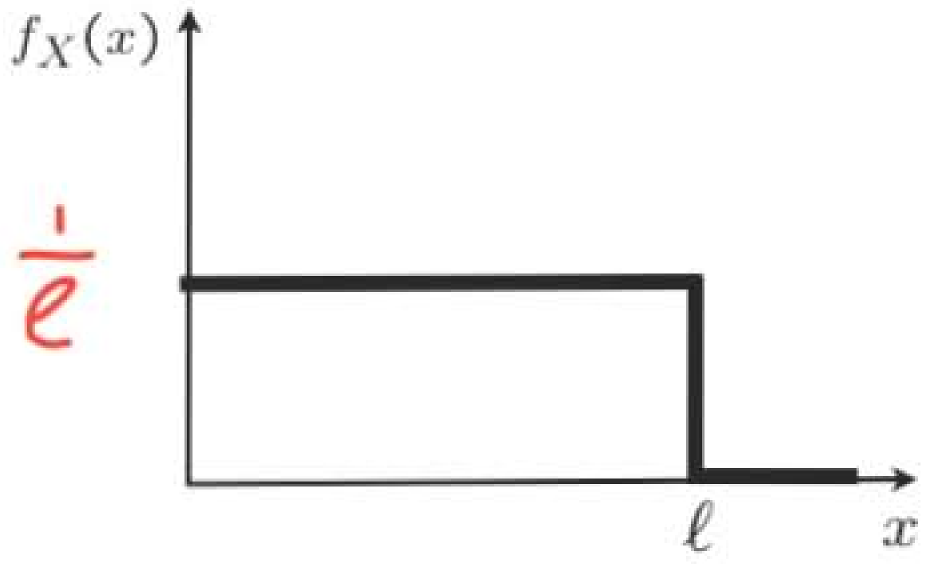
$\text{var}(X+Y) = \text{var}(X) + \text{var}(Y)$

$g(X)$ and $h(Y)$ are also independent: $\mathbf{E}[g(X)h(Y)] = \mathbf{E}[g(X)] \cdot \mathbf{E}[h(Y)]$

Stick-breaking example

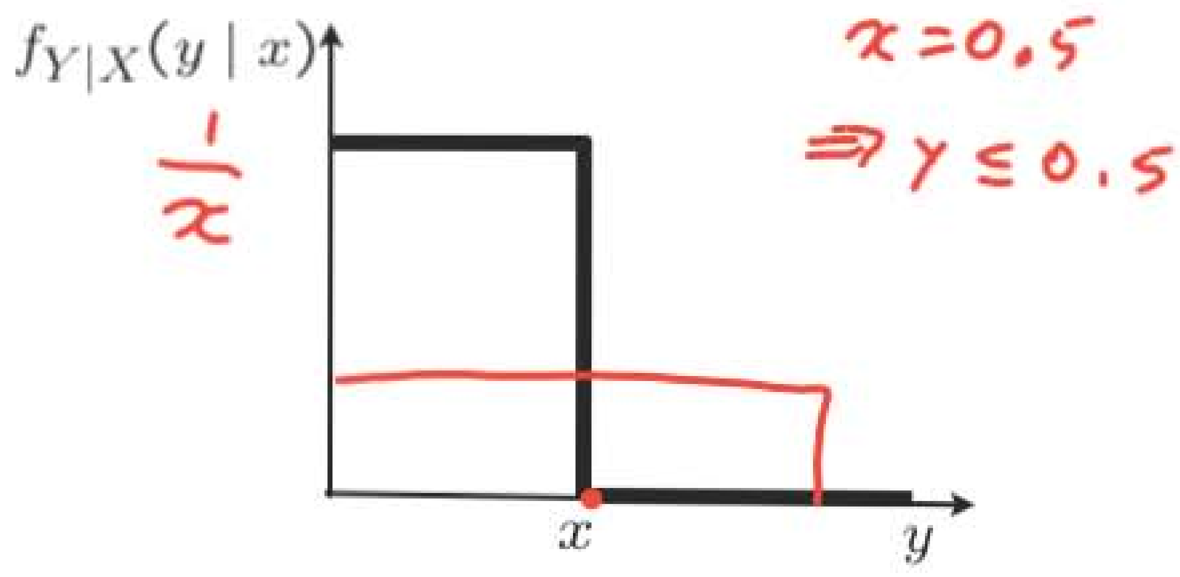


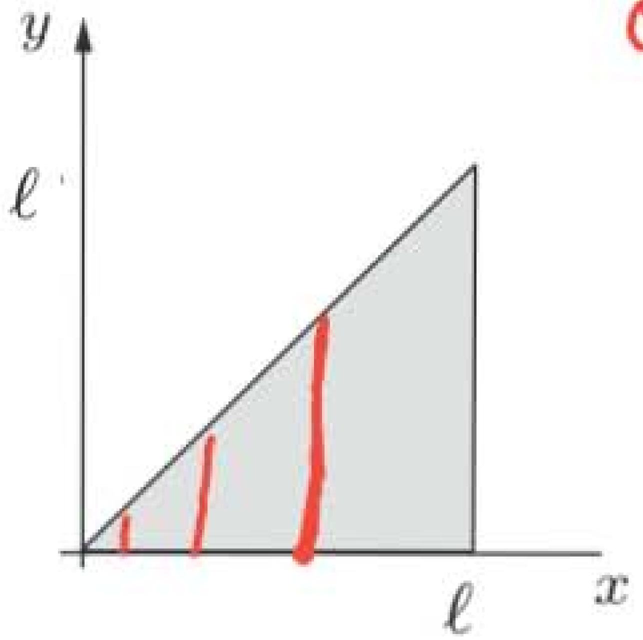
- Break a stick of length ℓ twice
  - first break at X: uniform in $[0, \ell]$
  - second break at Y: uniform in $[0, X]$



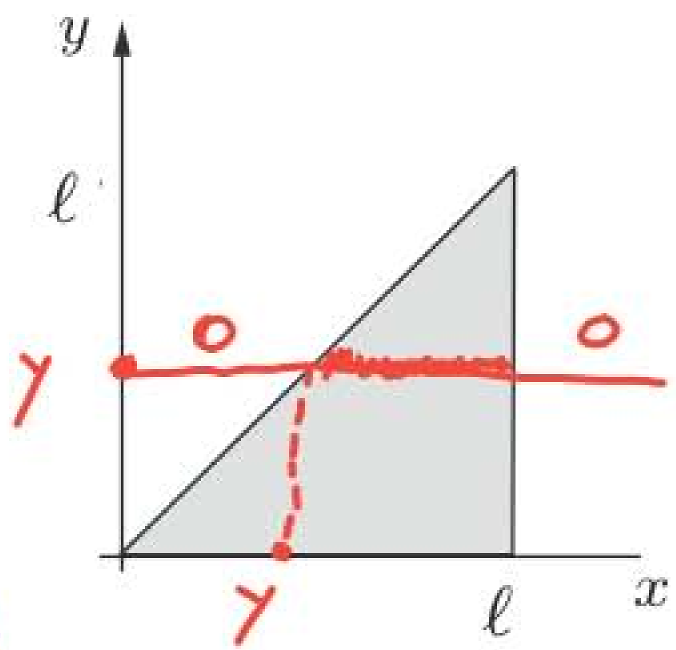
$$f_{X,Y}(x,y) = f_X(x) f_{Y|X}(y \mid x) = \frac{1}{\ell x}$$

$0 \leq y \leq x \leq \ell$



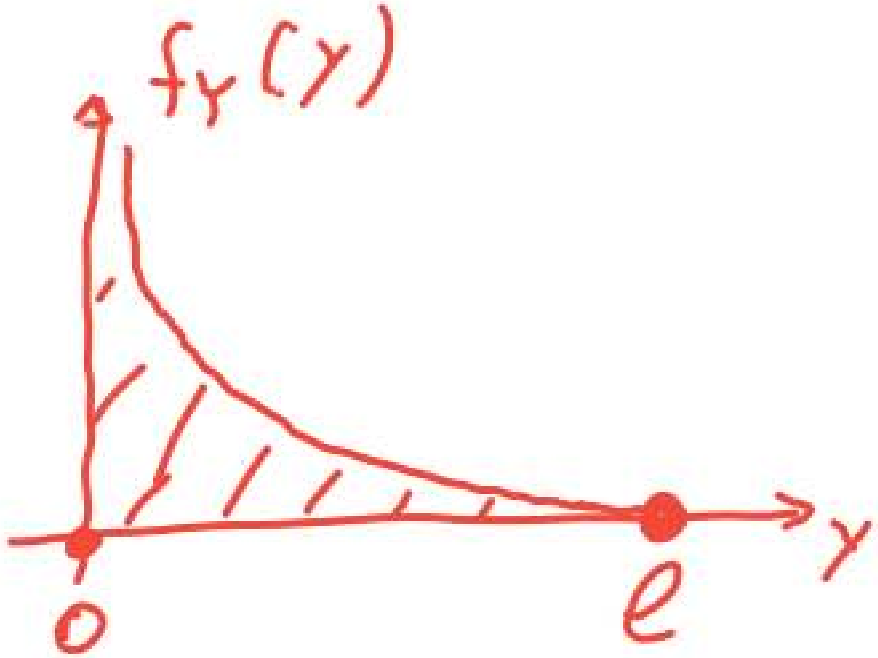


Stick-breaking example



$$f_{X,Y}(x,y) = \frac{1}{\ell x}, \qquad 0 \le y \le x \le \ell$$

$$f_Y(y) = \int f_{X,Y}(x,y)\,dx = \int_y^{\ell} \frac{1}{\ell x}\,dx = \frac{1}{\ell}\log\left(\frac{\ell}{y}\right)$$



$$\mathbf{E}[Y] = \int_0^{\ell} y\,\frac{1}{\ell}\log\left(\frac{\ell}{y}\right)dy$$

- Using total expectation theorem:

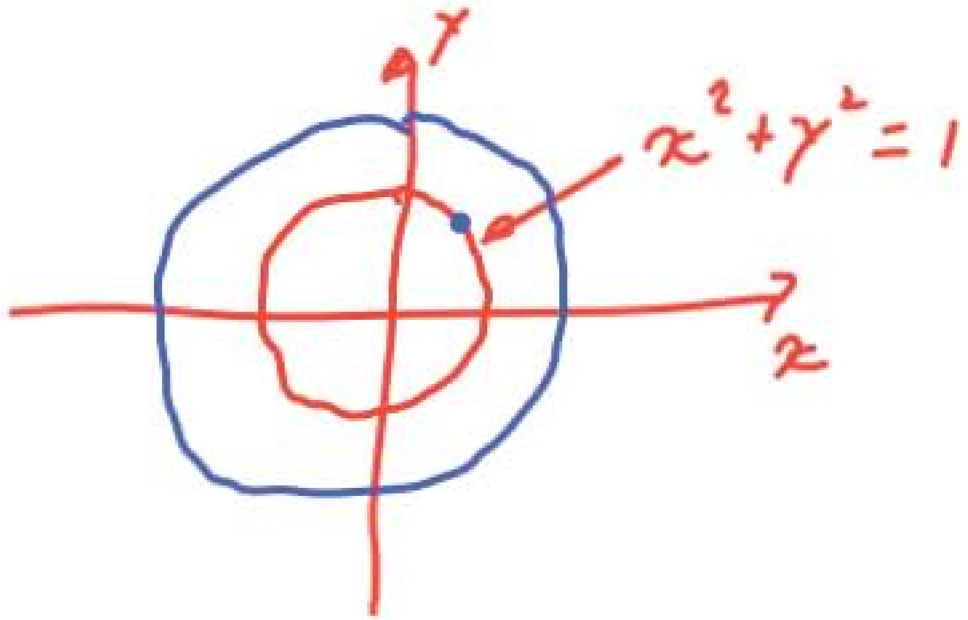
$$E[Y] = \int_0^{\ell} \frac{1}{\ell}\,E[Y \mid X = x]\,dx = \int_0^{\ell} \overbrace{\frac{1}{\ell}}^{f_X(x)}\,\frac{x}{2}\,dx = \frac{1}{2}E[X] = \frac{1}{2}\cdot\frac{\ell}{2} = \frac{\ell}{4}.$$

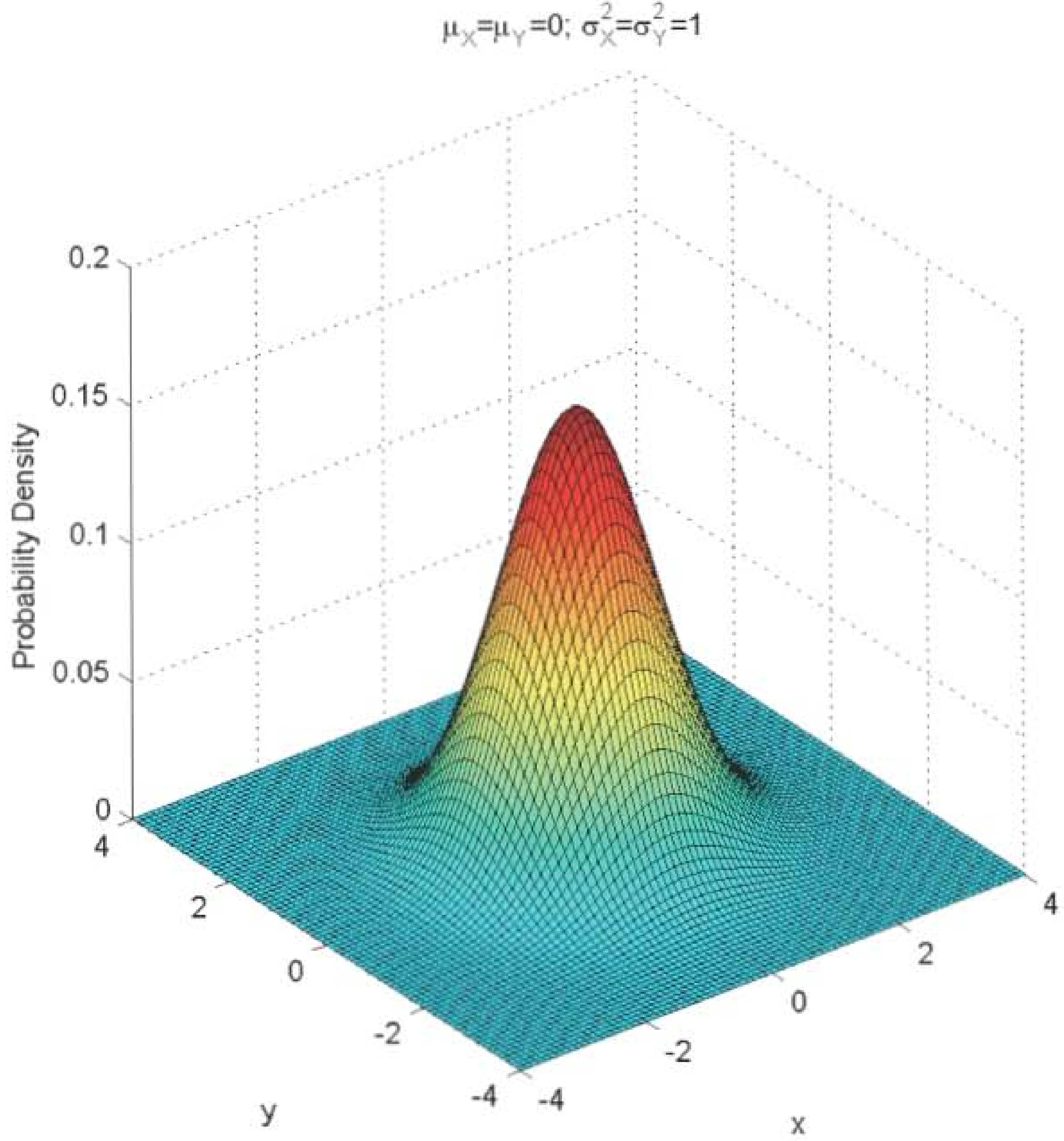
Independent standard normals

$$f_{X,Y}(x,y) = f_X(x) f_Y(y)$$

$$= \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{x^2}{2}\right\} \cdot \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{y^2}{2}\right\}$$

$$= \frac{1}{2\pi} \exp\left\{-\frac{1}{2}(x^2+y^2)\right\}$$

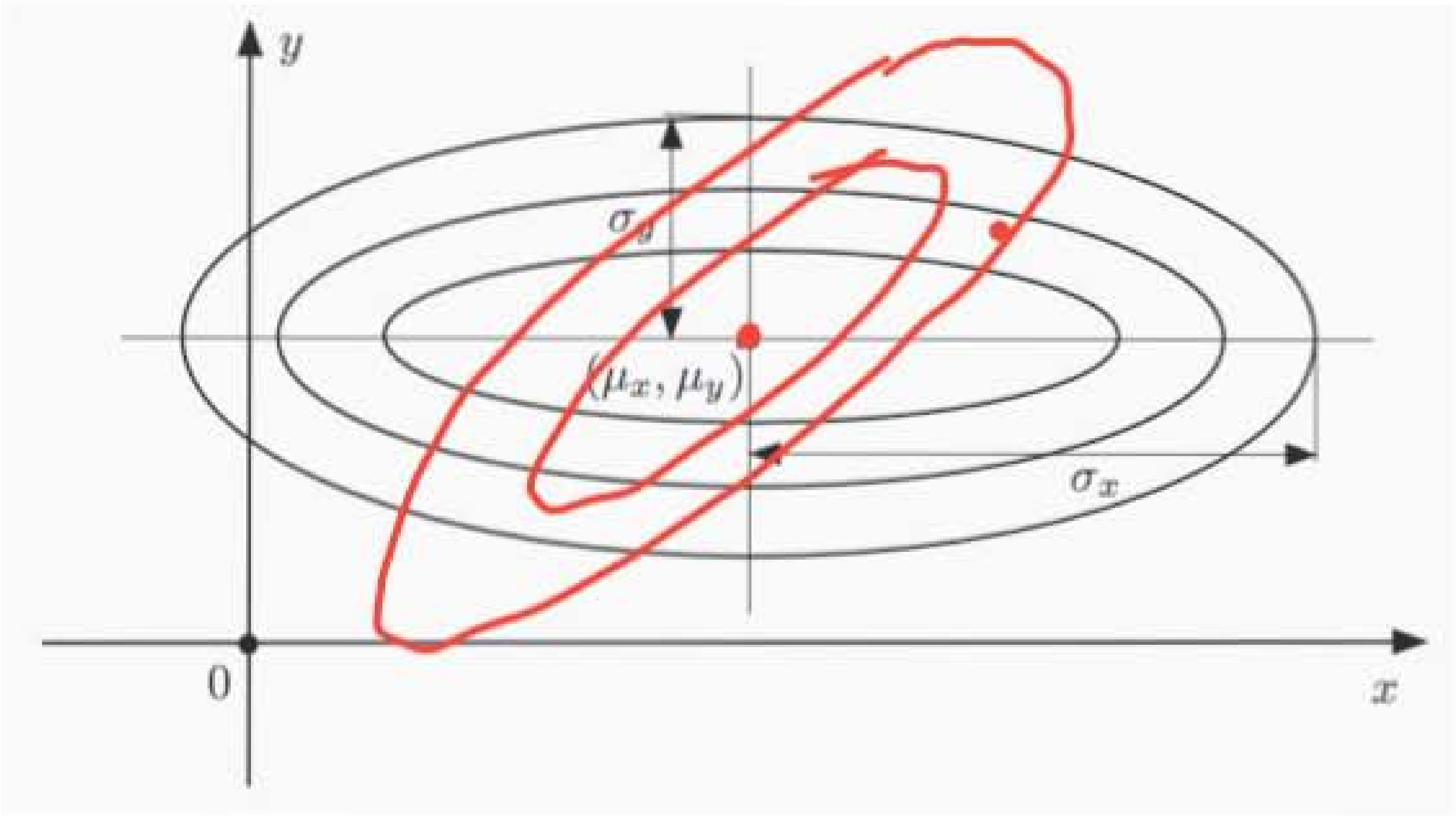


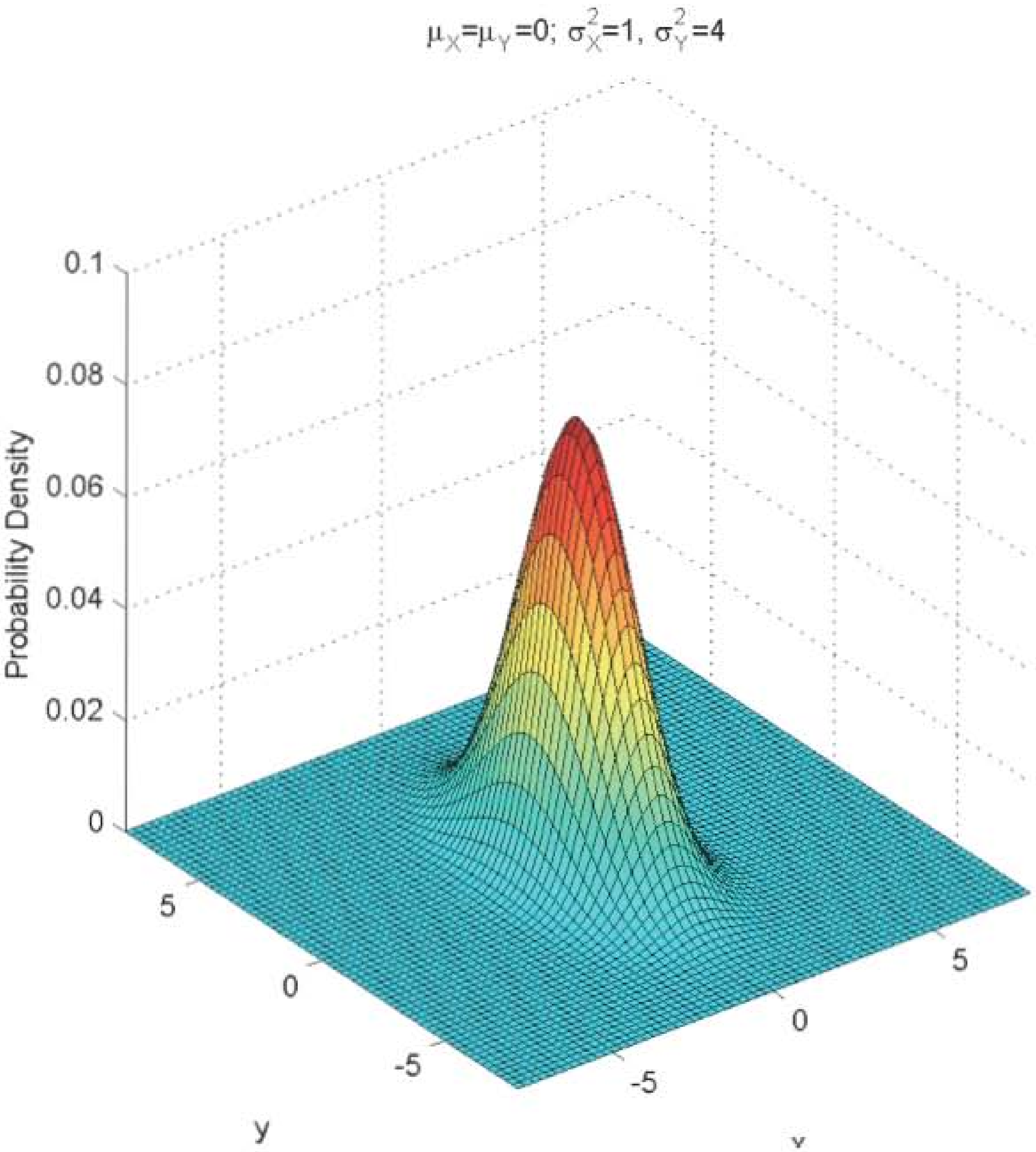


Independent normals

$$f_{X,Y}(x,y) = f_X(x) f_Y(y)$$

$$= \frac{1}{2\pi\sigma_x\sigma_y} \exp\left\{ -\frac{(x-\mu_x)^2}{2\sigma_x^2} - \frac{(y-\mu_y)^2}{2\sigma_y^2} \right\}$$





The Bayes rule — a theme with variations

$p_{X|Y}(\cdot \,|\, y)$

$$\begin{tikzcd}[column sep=huge] X \arrow[r] & Y \arrow[l, bend right, "\text{Inference}"'] \end{tikzcd}$$

unobserved value x | observed value y

prior $p_X(\cdot)$ | model $p_{Y|X}(\cdot \,|\, \cdot)$

$$\begin{aligned} p_{X,Y}(x,y) &= p_X(x)\, p_{Y|X}(y \,|\, x) \\ &= p_Y(y)\, p_{X|Y}(x \,|\, y) \end{aligned}$$

$$\begin{aligned} f_{X,Y}(x,y) &= f_X(x)\, f_{Y|X}(y \,|\, x) \\ &= f_Y(y)\, f_{X|Y}(x \,|\, y) \end{aligned}$$

$$p_{X|Y}(x \,|\, y) = \frac{p_X(x)\, p_{Y|X}(y \,|\, x)}{p_Y(y)}$$

$$f_{X|Y}(x \,|\, y) = \frac{f_X(x)\, f_{Y|X}(y \,|\, x)}{f_Y(y)}$$

posterior

$$p_Y(y) = \sum_{x'} p_X(x')\, p_{Y|X}(y \,|\, x')$$

$$f_Y(y) = \int f_X(x')\, f_{Y|X}(y \,|\, x')\, dx'$$

The Bayes rule — one discrete and one continuous random variable

K: discrete $\qquad\qquad$ Y: continuous

$\mathbf{P}(K=k,\ y \le Y \le y+\delta) \qquad \delta > 0,\ \delta \approx 0$

$= \mathbf{P}(K=k)\,\mathbf{P}(y \le Y \le y+\delta \mid K=k) \approx p_K(k)\, f_{Y|K}(y|k)\,\cancel{\delta}$

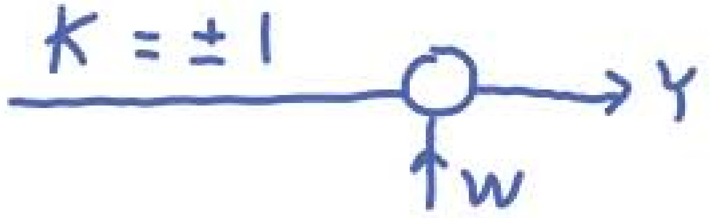
$= \mathbf{P}(y \le Y \le y+\delta)\,\mathbf{P}(K=k \mid y \le Y \le y+\delta) \approx f_Y(y)\,\cancel{\delta}\, p_{K|Y}(k|y)$

$$\boxed{p_{K|Y}(k \,|\, y) = \frac{p_K(k)\, f_{Y|K}(y \,|\, k)}{f_Y(y)}} \qquad \boxed{f_{Y|K}(y \,|\, k) = \frac{f_Y(y)\, p_{K|Y}(k \,|\, y)}{p_K(k)}}$$

$$f_Y(y) = \sum_{k'} p_K(k')\, f_{Y|K}(y \,|\, k') \qquad p_K(k) = \int f_Y(y')\, p_{K|Y}(k \,|\, y')\, dy'$$

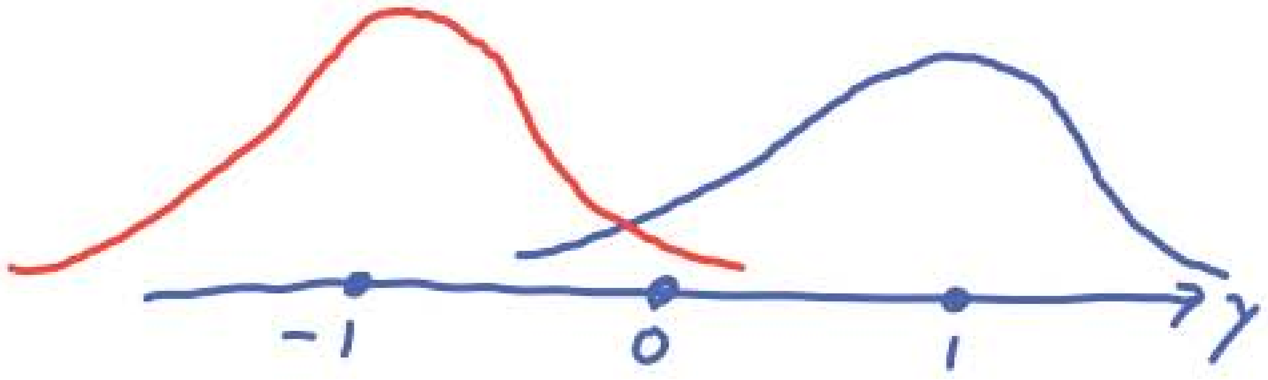
The Bayes rule — discrete unknown, continuous measurement

- unkown K: equally likely to be -1 or $+1$
- measurement Y: $Y = K + W$; $W \sim \mathcal{N}(0,1)$



$Y|K=1 \sim N(1,1)$

$Y|K=-1 \sim N(-1,1)$



- Probability that $K = 1$, given that $Y = y$? $p_{K|Y}(1|y)$

$$p_K(k) = 1/2 \qquad f_{Y|K}(y\,|\,k) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(y-k)^2}$$

$k = -1, +1$

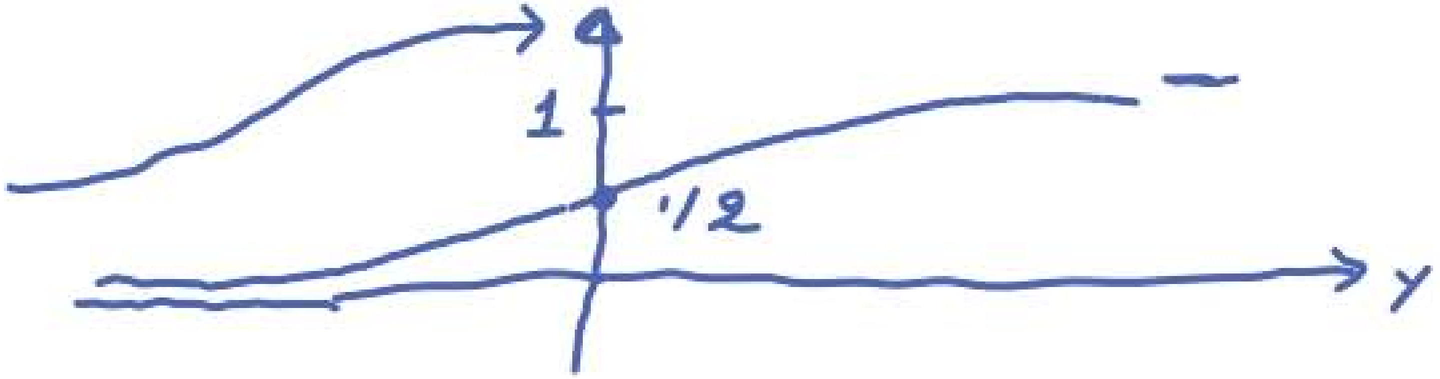
$$\boxed{p_{K|Y}(k\,|\,y) = \frac{p_K(k)\, f_{Y|K}(y\,|\,k)}{f_Y(y)}}$$

$$f_Y(y) = \frac{1}{2}\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(y+1)^2} + \frac{1}{2}\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(y-1)^2}$$

$$f_Y(y) = \sum_{k'} p_K(k')\, f_{Y|K}(y\,|\,k')$$

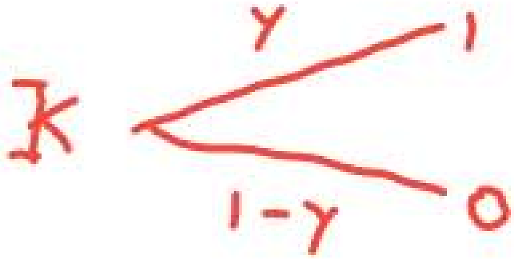
$$p_{K|Y}(1\,|\,y) = \frac{1}{1+e^{-2y}}$$

algebra



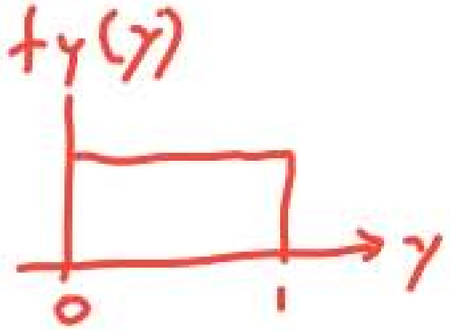
The Bayes rule — continuous unknown, discrete measurement

- measurement K: Bernoulli with parameter Y



$$\boxed{f_{Y|K}(y\,|\,\underset{\downarrow 1}{k}) = \frac{f_Y(y)\,p_{K|Y}(k\,|\,y)}{p_K(k)}}$$

- unkown Y: uniform on $[0,1]$



$$p_K(k) = \int f_Y(y')\,p_{K|Y}(k\,|\,y')\,dy'$$

- Distribution of Y given that $K=1$? $f_{Y|K}(y|1)$

$$f_Y(y) = \begin{cases} 1 & y \in [0,1] \\ 0 & \text{otherwise} \end{cases} \qquad p_{K|Y}(1\,|\,y) = y$$

$$p_K(1) = \int_0^1 1 \cdot y\,dy = \frac{y^2}{2}\Big|_0^1 = \frac{1}{2}$$

$$f_{Y|K}(y\,|\,1) = \frac{1 \cdot y}{1/2} = 2y\,,\quad y \in [0,1]$$

$f_{Y|K}(y|1)$