OK, so we have just seen that if we have a single recurrent class, which is not periodic, then the Markov chain reaches steady state, and the steady state probabilities satisfy the following system of equations.

These equations are essential in the study of Markov chains, and they have a name.

They are called the balance equations.

In fact, it's worth looking at them in a somewhat different way than how we introduced them so far.

Using a frequency interpretation.

Along the way, it will shed some light on the how or why they are called balance equations.

Intuitively, one can sometimes think of probabilities as frequencies.

For example, if we keep tossing a fair coin, which has a probability half of Heads, and then in the long run, half of the time we are going to see Heads.

So let us try an interpretation of this kind for pi of j, the steady state probability of state j.

Imagine the evolution of a Markov chain as a particle jumping from state to state.

And imagine an observer at a given state.

So imagine that you have an observer here, in a given state j.

And imagine that this observer will keep counting every time the particle visits the state j.

So you have this observer keeping track every time the particle is in state j and keep recording.

So, for example, one record at time two, and saw it at time four, eight, maybe n.

And you can look at the total number of time this observer saw the particle being in j, and you can define it.

Let's call it y of j of n.

So yj of n represents the total number of ones that you have up to time n.

So it's the total number, so we divide by n to have the frequency.

So here that would be the frequency of time this observer saw the particle in state j up to time n.

Well, when n is very, very large, so n large, that frequency approaches pi of j.

In fact, we can make it more rigorous by saying that that converges to pi of j when n goes to infinity in a rigorous fashion that we will not discuss here.

So we have now a frequency interpretation of pi of j.

Now, let us think about a frequency interpretation of transitions from 1 to j.

So again, you have a new observer, and this observer look at it here, and every time the particle pass here, he put a one.

So, for example, maybe one here and here.

So if you think about it, you're looking at from 1 to j, and of n, that would be the total number of ones that you observe here up to time n.

And if you divide by 1, that's the frequency, and so what is this frequency?

Well, let's look at it this way.

So how often do we have such a transition?

Well, a fraction pi1 of the time, the particle is in state 1, and whenever at state 1, there is going to be a probability p1j of going there.

There might be other ways to go, but out of all the time the particle is in state one, the frequency of time it will transition to j will be pi 1j.

So out of all possible transitions that can happen, the fraction of these transitions that will happen from 1 to j will be pi 1 times p1j.

Again, this is when n is large, and this can be made more rigorous.

Now, what's the total frequency of transitions into state j?

So these are transitions leaving.

These are the transitions of interest here.

So think about a third observer looking here and recording every time the particle goes through here, here, here, or here.

So what is the frequency of transition here?

Well, it will be the sum of all the possible transitions that we have observed there.

And so this is going to be this and that corresponds to this.

Now, the last step of the argument is to see that the particle is in state j, if and only if the last transition was into state j.

And this explains that this part, which we have calculated here, will be the same as this one and that explain that.

So this equation expresses exactly the statement that we made.

That's useful intuition to have, and we are going to see an example later on how it gives us shortcuts into analyzing Markov chains.